JOURNAL OF FUNCTIONAL ANALYSIS 82, 303–315 (1989)

Positive Harmonic Functions on Abelian Covers

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Communicated by R. B. Melrose

Received June 5, 1987

We prove the existence of positive harmonic functions and Green's functions on certain Abelian covers of non-compact, finite volume Riemann surfaces. These results are obtained by studying the asymptotic distribution of the lattice in hyperbolic space generated by the fundamental group of the covering surface. © 1989 Academic Press, Inc.

1. INTRODUCTION

In a recent paper of Lyons and Sullivan, the theory of harmonic functions on Riemann surfaces which arise as infinite, Abelian covers of compact Riemann surfaces was studied via the theory of Brownian motion. They showed that such a manifold never has a non-constant, positive harmonic function but does possess a Green's function if the rank of the covering group is three or more. They state that under weaker hypotheses, that include Abelian covers of finite volume Riemann surfaces, there are no non-constant bounded harmonic functions.

In this note we will extend these results to abelian covers of non-compact, finite area surfaces. In this case the results are somewhat different as the parabolic elements behave quite differently under covers than do classes arising from closed geodesics. Using Brownian motion, Lyons and McKean and McKean and Sullivan have treated the thrice and four times punctured spheres, see [5, 8]. They showed that on the Abelian cover of the thrice punctured sphere arising from the commutator subgroup there is a Green's function but no positive harmonic function, whereas the analogous cover of the four times punctured sphere has both a Green's function and positive harmonic functions. Theorem 2 was proved in somewhat greater generality by M. Rees. Her argument uses symbolic dynamics and ergodic theory; it does not require that the base manifold have finite volume, see [12]. Finally, N. Varopoulos has studied, via probability theory, the divergence of the Poincaré series at the critical exponent. His most recent paper [14], contains an extensive bibliography.

0022-1236/89 $3.00

Instead of Brownian motion we will use estimates for the asymptotic distribution of the lattice in hyperbolic space generated by the fundamental group of such a surface. These estimates are obtained from results on the spectral theory of the Laplace Beltrami operator proved in [2], by applying the Lax–Phillips wave equation method, see [6]. The necessary modifications of their method were carried out in [1]. These results easily extend to higher dimensional hyperbolic manifolds.

We will represent Riemann surfaces as quotients of the upper half plane, H^2 by discrete subgroups, Γ of $PSl(2, \mathbf{R})$. We will assume that Γ has no finite order elements. If Σ is a Riemann surface of genus g with $n+1>0$ punctures, then Γ has a presentation in terms of generators

$$\{\gamma_1, ..., \gamma_{2g}; p_1, ..., p_{n+1}\} \qquad \text{subject to the single relation}$$

$$\prod_{i=1}^{g} [\gamma_{2i-1}, \gamma_{2i}] \cdot \prod_{i=1}^{n+1} p_i = id. \tag{1.1}$$

The γ_i's are hyperbolic elements, whereas the p_i's are parabolic elements. Recall that Γ contains $(n+1)$-conjugacy classes of parabolic subgroups. Each such subgroup is conjugate in $Sl(2, \mathbf{R})$ to the group

$$G_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}.$$

The first homology group of Σ is isomorphic to $\mathbf{Z}^n \times \mathbf{Z}^{2g}$. Each homology class can be uniquely represented as

$$\prod_{i=1}^{n} p_i^{m_i} \prod_{i=1}^{2g} \gamma_i^{n_i} := [\mathbf{M}, \mathbf{N}] \in \mathbf{Z}^n \times \mathbf{Z}^{2g}. \tag{1.2}$$

Note that in this representation the homology class of the parabolic element p_{n+1} is $[(-1, ..., -1); 0]$.

The unitary representations, or characters of $H_1(\Sigma; \mathbf{Z})$ are parametrized by a $(n+2g)$-dimensional torus, $\mathbf{T}$. Let $(\xi_1, ..., \xi_n; \eta_1, ..., \eta_{2g})$ be linear coordinates for $\mathbf{T}$ which we define by

$$\begin{aligned} &\text{(a)} \quad \chi_{(\xi,\eta)}(\gamma_j) = e^{2\pi i \eta_j}, \qquad j = 1, ..., 2g; \\ &\text{(b)} \quad \chi_{(\xi,\eta)}(p_j) = e^{2\pi i \xi_j}, \qquad j = 1, ..., n. \end{aligned} \tag{1.3}$$

By virtue of (1.1) we must have

$$\chi_{(\xi,\eta)}(p_{n+1}) = e^{-2\pi i(\xi_1 + \cdots + \xi_n)}.$$

If Γ_1 is a normal subgroup of Γ with Γ/Γ_1 an abelian group without elements of finite order, then the group of deck transformations of H^2/Γ_1,

$\mathrm{Deck}(\Gamma; \Gamma_1)$ can be identified with a subgroup of $H_1(\Sigma; \mathbf{Z})$. Suppose that the rank of $\mathrm{Deck}(\Gamma; \Gamma_1)$ is k, then we can find generators of the form $\{(\tilde{\mathbf{M}}_1; \tilde{\mathbf{N}}_1), ..., (\tilde{\mathbf{M}}_k; \tilde{\mathbf{N}}_k)\}$. We define the p-rank of Γ_1, $\wp$ to be the real rank of the $n \times k$ matrix: $(\tilde{\mathbf{M}}_1, ..., \tilde{\mathbf{M}}_k)$. Using Gaussian elimination it is easy to show that we can replace the above basis with another integer spanning set $\{(\mathbf{M}_1; \mathbf{N}_1), ..., (\mathbf{M}_k; \mathbf{N}_k)\}$ such that

(1) $R\text{-Span}\{\tilde{\mathbf{M}}_1, ..., \tilde{\mathbf{M}}_k\} = R\text{-Span}\{\mathbf{M}_1, ..., \mathbf{M}_\wp\}$,

(2) $\mathbf{M}_{\wp+1} = \cdots = \mathbf{M}_k = 0$, and

(3) real $\mathrm{Span}\{(\tilde{\mathbf{M}}_1; \tilde{\mathbf{N}}_1), ..., (\tilde{\mathbf{M}}_k; \tilde{\mathbf{N}}_k)\} =$ real Span $\{(\mathbf{M}_1; \mathbf{N}_1), ..., (\mathbf{M}_k; \mathbf{N}_k)\}$.

To study the spectral theory of Δ on H^2/Γ_1 via the Floquet method, see [1], we need to determine the unitary representations of $\mathrm{Deck}(\Gamma; \Gamma_1)$. Let $\boldsymbol{\beta} = (\beta^1, ..., \beta^k)$; for $g \in \mathrm{Deck}(\Gamma, \Gamma_1)$ we define

$$\tilde{\chi}_{\boldsymbol{\beta}}(Lg) = \chi_{(\beta^l \mathbf{M}_l, \beta^l \mathbf{N}_l)}(g); \tag{1.4}$$

we use the summation convention on the right hand side.

Since the vectors $\{(\mathbf{M}_i, \mathbf{N}_i);\ i = 1, ..., k\}$ are a rational basis for $\mathrm{Deck}(\Gamma; \Gamma_1)$ we easily see that the different representations of $\mathrm{Deck}(\Gamma; \Gamma_1)$ are parametrized by the variable $\boldsymbol{\beta}$ modulo some lattice in $\mathbf{R}^k$. We will call this torus $\mathbf{T}_1$. The formula (1.4), essentially expresses $\mathbf{T}_1$ as a subgroup of $\mathbf{T}$.

The group Γ_1 is defined in terms of $\mathbf{T}_1$ by

$$\Gamma_1 = \bigcap_{\boldsymbol{\beta} \in \mathbf{T}_1} \ker \tilde{\chi}_{\boldsymbol{\beta}}. \tag{1.5}$$

To apply our construction of positive harmonic functions directly, it is necessary that Γ_1 contains at least one conjugacy class of parabolic subgroups. A little thought and formula (1.5) show that this is simply a question of whether $p_i \in \Gamma_1$ for some $i \in \{1, ..., n\}$. If the rank of Γ_1 is zero, then $\{p_1, ..., p_{n+1}\}$ are all in Γ_1. Besides this special case there is no general relation between the p-rank of Γ_1 and the number of the p_i which are in Γ_1. For example, the rank 1 cover of p-rank 1 with $\mathrm{Deck}(\Gamma; \Gamma_1)$ generated by $(1, ..., 1; 0)$ has no parabolic elements while that generated by $(0; 1, 0, ..., 0)$ contains all the elements $\{p_1, ..., p_{n+1}\}$.

Our results on harmonic functions follow by estimating

$$N_{\Gamma_1}(R) = \#\{\gamma \in \Gamma_1 : d(p, \gamma p) < R\}, \tag{1.6}$$

here p is a point in H^2 and $d(\cdot, \cdot)$ is the distance measured in the hyperbolic metric. We omit the explicit dependence on p as the asymptotic order of growth is independent of the point.

Our estimate is:

THEOREM 1. *If H^2/Γ is a Riemann surface of genus g with $(n+1)$-punctures so that $g=0$, $n \geqslant 2$ or $g \geqslant 1$, $n \geqslant 0$ and Γ_1 is normal subgroup with $\mathrm{Deck}(\Gamma; \Gamma_1)$ a torsion free Abelian group of rank k and p-rank $\wp$ then*

$$N_{\Gamma_1}(R) \sim \frac{e^R}{R^{(k+\wp)/2}}.$$

Here and in the sequel $A(R) \sim B(R)$ will mean that $A(R)$ is bounded above and below by fixed positive multiples of $B(R)$ as R tends to infinity. From this we conclude:

THEOREM 2. *The surface H^2/Γ_1 has a Green's function if and only if $k+\wp \geqslant 3$.*

And also

THEOREM 3. *If Γ_1 contains a non-trivial, conjugacy class of parabolic subgroups and $k+\wp \geqslant 3$ then H^2/Γ_1 has a non-constant positive harmonic function.*

To treat the case that Γ_1 has no parabolic elements one looks for a subgroup Γ_2 such that

(1) $\Gamma \rhd \Gamma_2 \rhd \Gamma_1$;

(2) $\Gamma_2/\Gamma_1 \simeq \mathbf{Z}$;

(3) Γ_2 contains a conjugacy class of parabolic elements;

(4) $k(\Gamma_2)+\wp(\Gamma_2) \geqslant 3$.

If such a subgroup can be found then Theorem 3 implies that there are non-constant, positive harmonic functions on H^2/Γ_2 which can then be lifted to H^2/Γ_1. This is similar in spirit to the two step procedure used in [7] to obtain a surface with a bounded holomorphic function; in that case Γ/Γ_1 is a solvable group. In the following cases the subgroup Γ_2 can always be found:

(a) p-rank $=1$ and $k \geqslant 4$; $n=1$

(b) p-rank $\geqslant 2$ and $k \geqslant 3$.

By combining our results with those of Lyons, McKean, and Sullivan we obtain a classification theorem for surfaces that arise from the commutator subgroup, $[\Gamma, \Gamma]$:

THEOREM 4. *If Γ is a fuchsian group of the first kind such that H^2/Γ is a surface of genus g with $n+1$ (>0) punctures then $H^2/(\Gamma, \Gamma)$:*

(i) *has a positive harmonic function but no bounded harmonic function if*

(a) $g=0$ *and* $n \geqslant 3$

(b) $g=1$ *and* $n \geqslant 2$

(c) $g \geqslant 2$ *and* $n \geqslant 0$

(ii) *has a Green's function but no positive harmonic function if* $g=0$ *and* $n=2$ *or* $g=1$ *and* $n=1$

(iii) *has no Green's function if* $g=1$ *and* $n=0$ *or* $g=0$ *and* $n \leqslant 1$.

The case $g=0$, $n \leqslant 1$ is classical; the fundamental group is abelian and the universal cover is C. The case $g=1$, $n=1$ follows using a small modification of the argument in [8].

2. LATTICE POINT ESTIMATE

In [1, 2] we studied the spectral theory of the Laplace–Beltrami operator, Δ on an hyperbolic manifold H^2/Γ_1, where Γ/Γ_1 is a free abelian group. In Section 1 we introduced $\mathbf{T}_1$ as the space of unitary characters of Γ/Γ_1. The spectral theory of H^2/Γ_1 is reduced to the study of L_β, an operator with symbol given by Δ and domain, the L^2-closure of Δ acting on smooth functions on H_2 which satisfy

$$f(\gamma z) = \tilde{\chi}_\beta(\gamma)\, f(z); \qquad \gamma \in \Gamma.$$

One shows that the Laplace operator on H^2/Γ_1 is unitarily equivalent to a direct integral

$$\Delta \simeq \int_{\beta \in T_1} L_\beta, \tag{2.1}$$

see [1, 11].

For a dense open subset of $\mathbf{T}_1$, L_β is an analytic family of operators, the complement of this set is a union of linear subspaces of $\mathbf{T}_1$, see Proposition 2.7 of [2]. For the application to counting lattice points all we need is the behavior of the spectrum of L_β for β near to zero, that is, for $\tilde{\chi}_\beta$ near to the trivial representation. In the aforementioned papers we have shown that whenever the p-rank of Γ_1 is non-zero the family of operators, L_β, fails to be analytic in any neighborhood of the trivial representation. Nonetheless, L_β is continuous in the strong resolvent sense, see [3], and we can find a continuous function $\lambda(\beta)$ and a continuous L^2-valued function $\phi(\beta)$ such that

$$\begin{aligned} &\text{(a)}\quad L_\beta\phi(\beta)+\lambda(\beta)\,\phi(\beta)=0\\ &\text{(b)}\quad \lim_{\beta\to 0}\lambda(\beta)=0 \\ &\text{(c)}\quad \int_{\mathscr{F}}|\phi(\beta)|^2\,dA=1;\ \lim_{\beta\to 0}\phi(\beta)=1/\sqrt{\mathrm{Vol}(\mathscr{F})}. \end{aligned} \tag{2.2}$$

Here $\mathscr{F}$ is a fundamental domain for the action of Γ on H^2, the limit in (c) is with respect to the L^2-norm; in virtue of (a) and (b) and the Sobolev embedding theorems, the limit is also locally uniform in the C^∞-topology.

There is moreover a $\lambda_0>0$ and a neighborhood N of zero in $\mathbf{T}_1$ such that

$$\sigma(\Delta)\cap[0,\lambda_0]=\{\lambda(\beta)\colon \beta\in N\}, \tag{2.3}$$

and so that the complete spectral projection of Δ onto the eigenspace $\mu\in[0,\lambda_0]$ is given by integration against $\phi(\beta)$;

$$dP_\mu f=\int_{\lambda(\beta)=\mu} f\overline{\phi(\beta)}\,dA.$$

If the p-rank is zero then the family of operators is analytic. The analysis in [4, 10] applies and one can prove the results in Section 1 rather easily. Henceforth we will assume the p-rank is at least 1. In [2] we derived estimates and an asymptotic formula for $\lambda(\xi,\eta)$. Setting

$$(\xi,\eta)=\sum_{l=1}^{k}\beta^l(\mathbf{M}_l;\mathbf{N}_l) \tag{2.4}$$

in these formulae we obtain estimates for $\lambda(\beta)$. We will state an asymptotic formula for $N_{\Gamma_1}(R)$ and give an outline of the derivation. The argument is very similar to that presented in [1, pp. 103–117] and the details are left to the reader.

The eigenfunction $\phi(p,\beta)$ is defined initially for p in a fundamental domain, $\mathscr{F}$, and is extended smoothly to H^2 by setting

$$\phi(\gamma p;\beta)=\tilde{\chi}_\beta(\gamma)\,\phi(p;\beta).$$

We define

$$\Sigma(R)=\sqrt{\pi}\int_N\frac{\Gamma(\sqrt{1/4-\lambda(\beta)})}{\Gamma(3/2+\sqrt{1/4-\lambda(\beta)})}\phi(p;\beta)\,\overline{\phi(p;\beta)} \times e^{[1/2+\sqrt{1/4-\lambda(\beta)}]R}\,d\beta_1\cdots d\beta_k. \tag{2.5}$$

Here N is the small neighborhood of zero in $\mathbf{T}_1$ introduced in (2.3). The method of Lax and Phillips as adapted in [1] gives the following result.

THEOREM 2.1. *There is a $\gamma < 1$ and a neighborhood N of zero in $\mathbf{T}_1$ such that*

$$N_{\Gamma_1}(R) = \Sigma(R) + O(e^{\gamma R}). \tag{2.6}$$

Here $\gamma = \min_{\beta \in N} (1/2 + \sqrt{1/4 - \lambda(\beta)})$.

The leading order behavior in (2.6) is not very explicit. As we need only the asymptotic order of growth we will discard the terms in (2.5) that are not needed to obtain

$$N_{\Gamma_1}(R) \sim \int_N e^{[1/2 + \sqrt{1/4 - \lambda(\beta)}]R} \, d\beta_1 \cdots d\beta_k. \tag{2.7}$$

To verify (2.7) one only need observe that $\phi(\beta)$ solves an elliptic equation (2.2)(a) which leads easily to pointwise upper bounds. A lower bound is obtained by using the observation made after (2.2) that $\phi(p, \beta)$ converges locally uniformly to $1/\sqrt{\mathrm{Vol}(\mathscr{F})}$. The other terms are trivially bounded above and below by positive constants.

Remark. A more careful analysis would lead to an exact asymptotic formula with a non-trivial error estimate, see [1].

To complete the derivation of the asymptotic formula and the proof of Theorem 1 we need to study the behavior of the integrand in (2.7) as $R \to \infty$. We replace $\sqrt{1/4 - \lambda(\beta)}$ with $1/2 - \lambda(\beta) + O((\lambda(\beta))^2)$ to obtain

$$N_{\Gamma_1}(R) \sim e^R \int_N e^{-(\lambda(\beta) + O((\lambda(\beta))^2))R} \, d\beta_1 \cdots d\beta_k. \tag{2.8}$$

This is the integral estimated in [2, Sect. 8]. In that paper we consider only the case $\Gamma_1 = [\Gamma, \Gamma]$, the commutator subgroup of Γ. However, one easily sees that the same analysis applies in the more general case considered here. In [2] we derive the following asymptotic formulae for $\lambda(\xi, \eta)$:

THEOREM A. *For any $\varepsilon > 0$ there exists positive constants, C_1, C_2, and ρ such that if*

$$|\xi| + |\eta| < \rho$$

then

$$C_1(|\eta|^2+|\xi|)\leqslant\lambda(\xi,\eta)\leqslant C_2(|\eta|^2+|\xi|^{1-\varepsilon}). \tag{2.9}$$

And

THEOREM B. *For $\varepsilon>0$ fixed and small we let*

$$D_\delta=\bigcap_{i=0}^{n}\{(\xi,\eta)\colon |\xi_i|\geqslant e^{-\delta/(|\eta|^2+|\xi|^{1-\varepsilon})}\},$$

with $\xi_0=\xi_1+\cdots+\xi_n$; and

$$B_{\rho_1,\rho_2}=\{(\xi,\eta)\colon |\xi|<\rho_1,\ |\eta|<\rho_2\},$$

then there exists $e(\delta,\rho_1,\rho_2)$ such that in $B_{\rho_1,\rho_2}\cap D_\delta$

$$\lambda(\xi,\eta)\simeq\frac{1}{2\operatorname{Vol}(\mathscr{F})}\sum_{i=0}^{n}|\xi_i|(1\pm e(\delta,\rho_1,\rho_2))$$
$$+\Sigma(a_{ij}\pm e(\delta,\rho_1,\rho_2)\,\delta_{ij})\eta_i\eta_j$$

$\xi_0=\xi_1+\cdots+\xi_n$.

Here a_{ij} is a positive definite matrix and $e(\delta,\rho_1,\rho_2)$ is a positive function that tends to zero as (δ,ρ_1,ρ_2) tends to $(0,0,0)$. The symbol "$\simeq$" indicates that the $+$ sign on the right is an upper bound while the $-$ sign is a lower bound. Recall that $(\xi,\eta)=\sum_{l=1}^{k}\beta^l(\mathbf{M}_l,\mathbf{N}_l)$. We have normalized so that $\mathbf{M}_j=0$ for $j=\wp+1,\ldots,k$; note that $\mathbf{M}_1,\ldots,\mathbf{M}_\wp$ are linearly independent as are $\mathbf{N}_{\wp+1},\ldots,\mathbf{N}_k$. Thus we see that are positive constants C_1 and C_2 such that

$$\begin{aligned}
&\text{(a)}\qquad C_1\sum_{j=1}^{\wp}|\beta_j|\leqslant|\xi|\leqslant C_2\sum_{j=1}^{\wp}|\beta_j|\\
&\text{(b)}\quad C_1\sum_{j=\wp+1}^{k}|\beta_j|^2\leqslant|\eta|^2\leqslant C_2\sum_{i=1}^{k}|\beta_j|^2.
\end{aligned}\tag{2.11}$$

In order to apply Theorem B we need a small modification in the definition of D_δ. This is because the linear space $\operatorname{Span}\{\mathbf{M}_1,\ldots,\mathbf{M}_\wp\}$ may be contained in a subspace of the form

$$\{\xi_{i_1}=\cdots=\xi_{i_m}=0\}.$$

The method used to derive (2.10) applies without modification to such subspaces, if we replace D_δ by

$$D'_\delta=\bigcap_{i\notin\{i_1,\ldots,i_m\}}\{(\xi,\eta)\colon|\xi_i|>e^{-\delta/(|\eta|^2+|\xi|^{1-\varepsilon})}\}\cap\{\xi_{i_1}=\cdots=\xi_{i_m}=0\}.$$

This follows easily from Proposition 2.7 of [2] and the method used to prove Theorem B.

Using the estimates, (2.11) in Theorems A and B, and the argument presented in [2, Sect. 8] we easily obtain that

$$\lim_{R \to \infty} \int_N e^{-R(\lambda(\beta) + O((\lambda(\beta))^{-2}))} \, d\beta_1 \cdots d\beta_k \sim 1/R^{(k+\wp)/2}. \tag{2.12}$$

In essence these estimates say that $\lambda(\beta)$ behaves like $\sum_{j=1}^{\wp} |\beta_j| + \sum_{j=\wp+1}^{k} |\beta_j|^2$, at least in most of a neighborhood of $\{\beta = 0\}$. The estimate (2.12) follows easily from this. Putting (2.11) into (2.8) we arrive at

$$N_{\Gamma_1}(R) \sim e^R / R^{(k+\wp)/2}, \tag{2.13}$$

which is the assertion of Theorem 1.

3. Applications to Function Theory

Using formula (2.12) we can now prove Theorems 2, 3, and 4. Let $d(x, y)$ be the hyperbolic distance between x and y in H^2. The Poincaré series for Γ_1 is defined by

$$f_1(x, y; s) = \sum_{\gamma \in \Gamma_1} e^{-s\, d(\gamma x, y)}. \tag{3.1}$$

The series converges for $\operatorname{Re} s > 1$. It is a classical fact that H^2/Γ_1 has a Green's function if and only if $f_1(x, y; 1)$ is finite, see [13]. It is also well known and easy to prove that if $f_1(x, y; 1)$ is finite for any pair of points (x, y) then it is finite for every pair, see [9]. We can rewrite the sum as a Lebesgue–Stieltjes integral

$$f_1(x, y; 1) = \int_c^{\infty} e^{-\lambda} \, dN_{\Gamma_1}(\lambda), \tag{3.2}$$

here c is a fixed positive constant for which $N_{\Gamma_1}(c) = 0$. Integrating by parts in (3.2) gives

$$f_1(x, x; 1) = e^{-\lambda} N_{\Gamma_1}(\lambda)\big|_c^{\infty} + \int_c^{\infty} e^{-\lambda} N_{\Gamma_1}(\lambda) \, d\lambda.$$

If we use the asymptotic formula (2.12) this becomes

$$f_1(x, x; 1) \sim \int_c^{\infty} e^{-\lambda} \cdot e^{\lambda} / \lambda^{(k+\wp)/2} \, d\lambda. \tag{3.3}$$

From (3.3) it follows that $f_1(x, x; 1) < \infty$ if and only if $k + \wp \geqslant 3$. This proves Theorem 2.

Proof of Theorem 3. To prove Theorem 3 we proceed in much the same way as one proceeds in the construction of the Eisenstein series. We use the upper half space model for H^2, so that Γ_1 is a subgroup of $PSl(2, \mathbf{R})$ which can be lifted to $Sl(2, \mathbf{R})$. The hypothesis that Γ_1 contains a non-trivial conjugacy class of parabolic subgroups implies that we can conjugate Γ_1 in $Sl(2, \mathbf{R})$ so that it contains

$$G_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\}.$$

Now, we consider the series

$$h(z; s) = \sum_{\{\gamma\} \varepsilon G_\infty \backslash \Gamma_1} (\operatorname{Im} z)^s. \tag{3.4}$$

We choose representatives $\{\gamma\}$ of $G_\infty \backslash \Gamma_1$ so that $0 < \operatorname{Re} \gamma z \leqslant 1$ if $0 < \operatorname{Re} z \leqslant 1$. It is clear that $h(z; s)$ is invariant under Γ_1. We will show that $h(z; 1)$ is finite if $f_1(z, z; 1)$ is finite. Again it suffices to consider any fixed z in H^2. That $h(z, 1)$ is comparable to $f_1(z, z, 1)$ follows from the formula for $d(z, \gamma z)$

$$d(z, \gamma z) = \cosh^{-1}\left(1 + \frac{|z - \gamma z|}{2 \operatorname{Im} z \operatorname{Im} \gamma z}\right).$$

If we fix z to be in $0 < \operatorname{Re} z \leqslant 1$ with $\operatorname{Im} z > 1$ and let $\operatorname{Im} \gamma z \to 0$ with $0 < \operatorname{Re} \gamma z \leqslant 1$ then

$$\begin{aligned} d(z, \gamma z) &\sim \log[|z - \gamma z| / \operatorname{Im} z \operatorname{Im} \gamma z] \\ &\sim \log \operatorname{Im} \gamma z. \end{aligned} \tag{3.5}$$

From (3.5) it follows that $h(z; s)$ is bounded by a constant times the series $f_1(z, z; s)$. Thus we see that if $k + \wp \geqslant 3$ then $h(z; 1)$ is an absolutely and locally uniformly convergent series. When $\operatorname{Re} s > 1$ we can differentiate (3.4) term by term to obtain

$$\Delta h + s(s-1)h = 0. \tag{3.6}$$

Integrating (3.6) against a function ϕ in $C_0^\infty(H^2)$ and integrating by parts it follows that

$$\int h \, \Delta\phi + s(s-1) \int h\phi = 0. \tag{3.7}$$

Since h is uniformly bounded on compact sets we can let s tend to one in (3.7) and obtain that $h(z; 1)$ is a weakly harmonic function. From Weyl's lemma it follows that $h(z; 1)$ is actually harmonic. It is clear that $h(z; 1)$ is positive as it is a sum of positive terms. It is non-constant as $h(z; 1) > \operatorname{Im} z$ and therefore $h(z; 1) \to \infty$ as $\operatorname{Im} z \to \infty$. This completes the proof of Theorem 3.

Proof of Theorem 4. Theorem 4 is a special case of Theorem 3. If the genus of H^2/Γ is 2 or more and there is at least one puncture then we can find a subgroup Γ_1 such that Γ/Γ_1 has rank 3 and Γ_1 has p-rank zero. From Theorem 3 it follows that H^2/Γ_1 has positive harmonic functions and therefore as $\Gamma_1 \rhd [\Gamma, \Gamma]$, so does $H^2/[\Gamma, \Gamma]$. In our representation of $\operatorname{Deck}(\Gamma; \Gamma_1)$ in $H_1(\Sigma)$ we could take

$$(\mathbf{M}_1; \mathbf{N}_1) = (0; 1, 0, 0,..., 0)$$
$$(\mathbf{M}_2; \mathbf{N}_2) = (0; 0, 1, 0,..., 0)$$
$$(\mathbf{M}_3; \mathbf{N}_3) = (0; 0, 0, 1, 0,..., 0).$$

If H^2/Γ has genus one and at least 3-cusps, we can find a subgroup Γ_1 such that Γ/Γ_1 has rank 3 and p-rank 1, and such that Γ_1 contains a non-trivial parabolic subgroup. For example, we could use $(1, 0; 0, 0)$, $(0, 0; 1, 0)$, and $(0, 0; 0, 1)$ as generators for $\operatorname{Deck}(\Gamma; \Gamma_1)$. Theorem 3 again applies to produce non-constant positive harmonic functions on H^2/Γ_1.

If the genus is 1 and there is a single cusp, then $H^2/[\Gamma, \Gamma]$ does not have a Green's function. If there are 2 cusps then $H^2/[\Gamma, \Gamma]$ has a Green's function. However, the argument of McKean and Sullivan shows there are no positive harmonic functions. Conformally H^2/Γ is equivalent to

$$C \backslash \{p + \Lambda\} \cup \{q + \Lambda\} / \Lambda,$$

where

$$\Lambda = \{m + n\tau : \operatorname{Im} \tau > 0; m, n \in \mathbf{Z}\};$$
$$p \not\equiv q \mod \Lambda.$$

We can construct $H^2/[\Gamma, \Gamma]$ in two steps:

(1) The planar surface $C \backslash \{p + \Lambda\} \cup \{q + \Lambda\}$ is a $\mathbf{Z} \oplus \mathbf{Z}$ cover.

(2) If $E(z)$ is an elliptic function on C/Λ with poles at p and q, with residues 1 and -1 then

$$w \to \left(w, \int_{w_0}^{w} E(z)\, dz\right)$$

defines a single valued conformal imbedding of $H^2/[\Gamma, \Gamma]$ into C^2. The covering group is the restriction of a group of affine transformations given by

$$g_1(w_1, w_2) = (w_1 + 1, w_2)$$
$$g_2(w_1, w_2) = (w_1 + \tau, w_2)$$
$$g_3(w_1, w_2) = (w_1, w_2 + 2\pi i).$$

Using the geometry of the imbedding it is easy to show that $d(g_3 p, p) \leqslant C$, where distance is measured in the hyperbolic metric on $H^2/[\Gamma, \Gamma]$. If $h(p)$ were a minimal positive harmonic function on $H^2/[\Gamma, \Gamma]$ then the Harnack inequality argument in [8] would show that

$$h(g_3 \cdot p) = h(p).$$

Thus $h(p)$ would actually be a positive harmonic function on $C \backslash \{p + \Lambda\} \cup \{q + \Lambda\}$. It is a classical fact that no such function exists.

The final case is the sphere. If there are four punctures then we can find a subgroup, Γ_1, with rank 2, p-rank 2 and two conjugacy classes of parabolic subgroups; e.g., we could take $\mathrm{Deck}(\Gamma; \Gamma_1)$ to be generated by $(1, 0, 0)$ and $(0, 1, 0)$. Theorem 3 assures the existence of non-constant positive harmonic functions. If there are three punctures then $H^2/[\Gamma, \Gamma]$ has a Green's function. In [5, 8] it is shown that this surface carries no positive harmonic functions.

The fact that $H^2/[\Gamma, \Gamma]$ carries no bounded harmonic functions follows from the work of Sullivan and Lyons. This is because the geodesic flow on H^2/Γ is recurrent whenever the area is finite. In the language of the classification theory of Riemann surfaces we have found that the surfaces $H^2/[\Gamma, \Gamma]$ are in $O_{HB} \backslash O_{HP}$. The existence of such surfaces was established by Ahlfors and Toki using a complicated and ingenious construction. From our results and those of Sullivan and Lyons we see that $O_{HB} \backslash O_{HP}$ is a large set containing most nice abelian covers of non-compact, finite area surfaces.

ACKNOWLEDGMENTS

I thank E. Calabi for suggesting that I consider this question and C. Series for pointing out the work of Lyons and Sullivan and Varopoulos.

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